Pyramid and Quantum Metric Measure Space

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博　士　論　文

Pyramid
and
Quantum Metric Measure Space

(ピラミッドと量子測度距離空間)

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Pyramid
and
Quantum Metric Measure Space

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A summary of the thesis

Gromov [4, Chapter $3\frac{1}{2}_+$] introduced two distance functions on the set, say $\mathcal{X}$, of isomorphism classes of mm-spaces. In this thesis, $X = (X, d_X, \mu_X)$ is called an *mm-space* (*metric measure space*) if (X, d_X) is a complete separable metric space and if μ_X is a Borel probability measure on X. One of Gromov's distance functions is the *λ-box distance function* $\square_\lambda$, $\lambda \geq 0$. This is a distance function on $\mathcal{X}$ and a generalization of the Prohorov distance between two Borel probability measures on a complete separable metric space. The other is the *observable distance function* d_{conc}. That comes from the idea of measure concentration phenomenon, and is defined by the difference between the sets of 1-Lipschitz functions on two mm-spaces. The two distance functions satisfy $d_{\mathrm{conc}} \leq \square_1$. In this thesis, we consider the asymptotic behavior of sequences of quantum metric measure spaces and pyramids. A quantum metric measure space is an element of a natural compactification of $(\mathcal{X}_1, \square_1)$, where $\mathcal{X}_1$ is the set of isomorphism classes of mm-spaces with diameter at most one (see Definition 18). A pyramid is an element of a natural compactification of $(\mathcal{X}, d_{\mathrm{conc}})$ (see Definition 1).

The measure concentration phenomenon is stated as that any 1-Lipschitz function on an mm-space is close to a constant function on a Borel set with almost full measure. Historically, this phenomenon was first discovered by Lévy [7]. He proved the measure concentration phenomenon on high-dimensional unit spheres $S^n(1)$. Milman [9] applied the measure concentration phenomenon for a proof of Dvoretzky's theorem. After that, many applications of measure concentration phenomenon to geometry, analysis, probability, and discrete mathematics were found. We consider an mm-space X to be close to one-point mm-space $* := (\{p\}, \delta_p)$ if for any 1-Lipschitz function on X is close to a constant function. Gromov generalized this idea and then introduced the observable distance function d_{conc}. A pyramid is defined as follows. For two mm-spaces X and X', we define $X' \prec X$ if there exists a 1-Lipschitz map $f : X \to X'$ such that $\mu_{X'} = f_*\mu_X$.

Definition 1 (Pyramid; Gromov, Shioya [4,14,15]). A subset $\mathcal{P} \subset \mathcal{X}$ is called a *pyramid* if it satisfies the following (1), (2), and (3).

(1) If $X \in \mathcal{P}$ and if $X' \prec X$, then $X' \in \mathcal{P}$.

(2) For any $X, X' \in \mathcal{P}$, there exists $Z \in \mathcal{P}$ such that $X \prec Z$ and $X' \prec Z$.

(3) $\mathcal{P}$ is a non-empty $\square$-closed set.

We denote the set of pyramids by Π.

In Gromov's book [4, Section $3\frac{1}{2}.51$], the definition of a pyramid is given only by (1) and (2) of Definition 1. The condition (3) is added by Shioya [14,15]. For an mm-space X, we define

$$\mathcal{P}_X := \{\ X' \in \mathcal{X} \mid X' \prec X\ \}.$$

Then $\mathcal{P}_X$ is a pyramid. A map $\iota : \mathcal{X} \ni X \mapsto \mathcal{P}_X \in \Pi$ is injective. There are two special pyramids. One is the minimal pyramid $\mathcal{P}_* = \{*\}$ with respect to the inclusion relation. The other is the maximal pyramid $\mathcal{X}$ with respect to the inclusion relation. Gromov introduced the *weak convergence* of pyramids and claimed the following (1)–(3) in [4, Section $3\frac{1}{2}$.51-52].

(1) The embedding map $\iota : \mathcal{X} \to \Pi$ is a topological embedding map with respect to d_{conc} and the weak convergence.

(2) The image $\iota(\mathcal{X})$ is dense on Π with respect to the weak convergence.

(3) Π is sequentially compact with respect to the weak convergence.

Shioya [14, 15] defined a distance function ρ on Π compatible with the weak convergence of pyramids. He proved the embedding map ι is a 1-Lipschitz topological embedding map with respect to d_{conc} and ρ. Then (Π, ρ) is a compactification of $(\mathcal{X}, d_{\mathrm{conc}})$. In particular, (Π, ρ) is a compact metric space.

There are fundamental invariants called the *observable diameter* $\mathrm{ObsDiam}(X; -\kappa)$, the *separation distance* $\mathrm{Sep}(X; \kappa_0, \kappa_1, \ldots, \kappa_N)$, and the *concentration function* $\alpha_X(r, \kappa)$. They are related to the measure concentration phenomenon. The observable diameter and the separation distance are introduced by Gromov [4, Chapter $3\frac{1}{2}{}_+$]. The concentration function is introduced by Amir-Milman [1]. Let X be an mm-space and $r, \kappa, \kappa_0, \kappa_1, \ldots, \kappa_N$ positive real numbers with $N \geq 1$.

$$
\begin{aligned}
&\mathrm{ObsDiam}(X; -\kappa) \\
&:= \sup_{f \in \mathcal{L}ip_1(X)} \inf\{\, \mathrm{diam}(A) \mid A \subset \mathbb{R} : \text{Borel set},\ \mu_X(f^{-1}(A)) \geq 1 - \kappa \,\}, \\
&\mathrm{Sep}(X; \kappa_0, \kappa_1, \ldots, \kappa_N) \\
&:= \sup\{\, \min_{i \neq j} d_X(A_i, A_j) \mid A_i \subset X : \text{Borel set},\ \mu_X(A_i) \geq \kappa_i,\ i = 0, 1, \ldots, N \,\}, \\
&\alpha_X(r, \kappa) \\
&:= \sup\{\, 1 - \mu_X(U_r(A)) \mid A \subset X : \text{Borel set},\ \mu_X(A) \geq \kappa \,\},
\end{aligned}
$$

where $\mathcal{L}ip_1(X)$ is the set of 1-Lipschitz functions on X and $U_r(A)$ is an open r-neighborhood of A.

The observable diameter and the concentration function express the closeness between X and $*$ with respect to d_{conc} (see Proposition 6). The separation distance expresses the closeness between X and an mm-space consisting of at most N points with respect to d_{conc} (see Theorem 8 and Corollary 9). The author and Shioya together generalized these invariants for pyramid and proved the following limit formulas.

Theorem 2 (Ozawa-Shioya [12]). *Let $\mathcal{P}$ and $\mathcal{P}_n$, $n = 1, 2, \ldots$, be pyramids. If $\mathcal{P}_n$ converges weakly to $\mathcal{P}$ as $n \to \infty$, then*

$$
\begin{aligned}
\mathrm{ObsDiam}(\mathcal{P}; -\kappa) &= \lim_{\delta \to 0+} \liminf_{n \to \infty} \mathrm{ObsDiam}(\mathcal{P}_n; -(\kappa + \delta)) \\
&= \lim_{\delta \to 0+} \limsup_{n \to \infty} \mathrm{ObsDiam}(\mathcal{P}_n; -(\kappa + \delta))
\end{aligned}
$$

for any positive real number κ.

Theorem 3 (Ozawa-Shioya [12]). *Let $\mathcal{P}$ and $\mathcal{P}_n$, $n = 1, 2, \ldots$, be pyramids. If $\mathcal{P}_n$ converges weakly to $\mathcal{P}$ as $n \to \infty$, then*

$$\begin{aligned}\mathrm{Sep}(\mathcal{P}; \kappa_0, \kappa_1, \ldots, \kappa_N) &= \lim_{\delta \to 0+} \liminf_{n \to \infty} \mathrm{Sep}(\mathcal{P}_n; \kappa_0 - \delta, \kappa_1 - \delta, \ldots, \kappa_N - \delta) \\ &= \lim_{\delta \to 0+} \limsup_{n \to \infty} \mathrm{Sep}(\mathcal{P}_n; \kappa_0 - \delta, \kappa_1 - \delta, \ldots, \kappa_N - \delta)\end{aligned}$$

for any positive real numbers $\kappa_0, \kappa_1, \ldots, \kappa_N$ with $N \geq 1$.

Theorem 4 (Ozawa [11]). *Let $\mathcal{P}$ and $\mathcal{P}_n$, $n = 1, 2, \ldots$, be pyramids. If $\mathcal{P}_n$ converges weakly to $\mathcal{P}$ as $n \to \infty$, then*

$$\begin{aligned}\alpha_{\mathcal{P}}(r, \kappa) &= \lim_{\delta \to 0+} \liminf_{n \to \infty} \alpha_{\mathcal{P}_n}(r - \delta, \kappa - \delta) \\ &= \lim_{\delta \to 0+} \limsup_{n \to \infty} \alpha_{\mathcal{P}_n}(r - \delta, \kappa - \delta)\end{aligned}$$

for any $0 < \kappa \leq 1$ and $r > 0$.

As a first application of these limit formulas, the author and Shioya together proved the following (1) and (2).

(1) The observable diameter for pyramid expresses the closeness between a pyramid and $\mathcal{P}_*$ (see Proposition 6).

(2) The separation distance for pyramid expresses the closeness between a pyramid and an extended mm-space consisting of at most N points with respect to ρ, where 'extended' means that the distance between two points is allowed to be infinity (see Theorem 8).

Definition 5 (Lévy family). A sequence of pyramids $\{\mathcal{P}_n\}_{n=1}^{\infty}$ is called a *Lévy family* if

$$\lim_{n \to \infty} \mathrm{ObsDiam}(\mathcal{P}_n; -\kappa) = 0$$

for any positive real number κ. A sequence of mm-spaces $\{X_n\}_{n=1}^{\infty}$ is called a *Lévy family* if so is $\{\mathcal{P}_{X_n}\}_{n=1}^{\infty}$

Proposition 6 (Ozawa-Shioya [12]). *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of pyramids. Then the following* (1) *and* (2) *are equivalent to each other.*

(1) *$\{\mathcal{P}_n\}_{n=1}^{\infty}$ is a Lévy family.*

(2) *$\mathcal{P}_n$ converges weakly to $\mathcal{P}_*$ as $n \to \infty$.*

Definition 7 (N-Lévy family). Let N be a natural number. A sequence $\{\mathcal{P}_n\}_{n=1}^{\infty}$ of pyramids is called an *N-Lévy family* if

$$\lim_{n \to \infty} \mathrm{Sep}(\mathcal{P}_n; \kappa_0, \kappa_1, \ldots, \kappa_N) = 0$$

for any $\kappa_0, \kappa_1, \ldots, \kappa_N$ with $\sum_{i=0}^{N} \kappa_i < 1$. A sequence of mm-spaces $\{X_n\}_{n=1}^{\infty}$ is called an *N-Lévy family* if so is $\{\mathcal{P}_{X_n}\}_{n=1}^{\infty}$

Note that a 1-Lévy family coincides with a Lévy family.

Theorem 8 (Ozawa-Shioya [12]). *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of pyramids converging weakly to a pyramid $\mathcal{P}$, and N a natural number. Then the following* (1) *and* (2) *are equivalent to each other.*

(1) *$\{\mathcal{P}_n\}_{n=1}^{\infty}$ is an N-Lévy family.*

(2) *There exists a finite extended mm-space Y consisting of at most N points such that $\mathcal{P} = \mathcal{P}_Y$.*

For a pyramid $\mathcal{P}$ and a positive real number t, we define $t\mathcal{P} := \{ \ tX \mid X \in \mathcal{P} \ \} \subset \mathcal{X}$, where $tX := (X, td_X, \mu_X)$. Then $t\mathcal{P}$ is also a pyramid.

Corollary 9 (Funano-Shioya, Ozawa-Shioya [3, 12]). *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be an N-Lévy family of pyramids such that*

$$\mathrm{ObsDiam}(\mathcal{P}_n; -\kappa) < +\infty$$

for any real number κ with $0 < \kappa < 1$ and for any natural number n. Then, we have one of the following (1) *and* (2).

(1) *$\{\mathcal{P}_n\}_{n=1}^{\infty}$ is a Lévy family.*

(2) *There is a subsequence $\{\mathcal{P}_{m(n)}\}_{n=1}^{\infty}$ of $\{\mathcal{P}_n\}_{n=1}^{\infty}$ and a sequence of real numbers $\{t_n\}_{n=1}^{\infty}$ with $0 < t_n \leq 1$ such that $t_n\mathcal{P}_{m(n)}$ converges weakly to $\mathcal{P}_X$ as $n \to \infty$ for some finite mm-space X with $2 \leq \#X \leq N$.*

This corollary is a generalization of [3, Theorem 4.4] and one of keys to prove a dimension-free estimate of the ratio of the N-th to the first eigenvalues of Laplacian on a closed Riemannian manifold with nonnegative Ricci curvature (see [3, Theorem 1.1]).

We consider the ∞-dissipation property of a sequence of pyramids. This is opposite from measure concentration phenomenon. A sequence of pyramids $\{\mathcal{P}_n\}_{n=1}^{\infty}$ ∞-*dissipates* if

$$\lim_{n\to\infty} \mathrm{Sep}(\mathcal{P}_n; \kappa_0, \kappa_1, \ldots, \kappa_N) = +\infty$$

for any positive real numbers $\kappa_0, \kappa_1, \ldots, \kappa_N$ with $N \geq 1$ and $\sum_{i=0}^{N} \kappa_i < 1$. $\{\mathcal{P}_n\}_{n=1}^{\infty}$ ∞-dissipates if and only if $\mathcal{P}_n$ converges weakly to $\mathcal{X}$ as $n \to \infty$. Let $S^n(r_n)$ be the sphere of radius r_n centered at the origin in the $(n+1)$-dimensional Euclidean space and equipped with the Euclidean distance function and the normalized Riemannian volume measure. Gromov-Milman [5] proved that $\{S^n(r_n)\}_{n=1}^{\infty}$ is a Lévy family if and only if $r_n/\sqrt{n} \to 0$ as $n \to \infty$. In particular, $\{\mathcal{P}_{S^n(r_n)}\}_{n=1}^{\infty}$ is a Lévy family if and only if $r_n/\sqrt{n} \to 0$ as $n \to \infty$. Shioya [14, 15] proved that $\{\mathcal{P}_{S^n(r_n)}\}_{n=1}^{\infty}$ ∞-dissipates if and only if $r_n/\sqrt{n} \to \infty$ as $n \to \infty$. Moreover, he proved that $\mathcal{P}_{S^n(\sqrt{n})}$ converges weakly to the pyramid which expresses the infinite-dimensional Gaussian space as $n \to \infty$. We consider a sequence of pyramids with the same property as for the sequence of spheres.

Definition 10 (Phase transition property; Ozawa-Shioya [12]). Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of pyramids. We say that $\{\mathcal{P}_n\}_{n=1}^{\infty}$ has the *phase transition property* if there exists a sequence of positive real numbers $\{c_n\}_{n=1}^{\infty}$ satisfying the following (1) and (2).

(1) For any sequence of positive real numbers $\{t_n\}_{n=1}^{\infty}$ with $t_n/c_n \to 0$ as $n \to \infty$, the sequence $\{t_n \mathcal{P}_n\}_{n=1}^{\infty}$ is a Lévy family.

(2) For any sequence of positive real numbers $\{t_n\}_{n=1}^{\infty}$ with $t_n/c_n \to +\infty$ as $n \to \infty$, the sequence $\{t_n \mathcal{P}_n\}_{n=1}^{\infty}$ ∞-dissipates.

We call such a sequence $\{c_n\}_{n=1}^{\infty}$ a sequence of *critical scale order*. We say that a sequence of mm-spaces $\{X_n\}_{n=1}^{\infty}$ has the *phase transition property* if so does $\{\mathcal{P}_{X_n}\}_{n=1}^{\infty}$.

As a second application of the limit formulas of observable diameter and separation distance, the author and Shioya together had a necessary and sufficient condition for a sequence of pyramids having the phase transition property.

Theorem 11 (Ozawa-Shioya [12])**.** *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of pyramids. Then the following* (1) *and* (2) *are equivalent to each other.*

(1) *$\{\mathcal{P}_n\}_{n=1}^{\infty}$ has the phase transition property.*

(2) *There exists a sequence $\{r_n\}_{n=1}^{\infty}$ of positive real numbers such that*

$$\mathrm{ObsDiam}(\mathcal{P}_n; -\kappa) \sim r_n$$

for any $0 < \kappa < 1$, where $a_n \sim b_n$ means that the ratios a_n/b_n and b_n/a_n for all n are bounded.

In this case, $\{1/r_n\}_{n=1}^{\infty}$ is a sequence of critical scale order.

Moreover, we found the following sequences of closed Riemannian manifolds having the phase transition property. The following sequences of Riemannian manifolds have the normalized Riemannian volume measure and then we consider it as an mm-space.

Corollary 12 (Ozawa-Shioya [12])**.** *The following sequences of Riemannian manifolds have the phase transition property with critical scale order $\sim \sqrt{n}$.*

(1) *The unit sphere $S^n(1)$.*

(2) *The complex projective space $\mathbb{C}P^n$.*

(3) *The quaternionic projective space $\mathbb{H}P^n$.*

(4) *The special orthogonal group $SO(n)$.*

(5) *The special unitary group $SU(n)$.*

(6) *The compact symplectic group $Sp(n)$.*

(7) *The real Stiefel manifold $\mathrm{St}_{k(n)}(\mathbb{R}^n)$ with $1 \leq k(n) < n$.*

(8) *The complex Stiefel manifold $\mathrm{St}_{k(n)}(\mathbb{C}^n)$ with $1 \leq k(n) < n$.*

(9) *The quaternionic Stiefel manifold $\mathrm{St}_{k(n)}(\mathbb{H}^n)$ with $1 \leq k(n) < n$.*

We consider the unit sphere $S_p^n(1)$ in $(\mathbb{R}^{n+1}, \|\cdot\|_p)$. The unit sphere $S_p^n(1)$ has normalized Hausdorff measure $\mu_{S_p^n(1)}$.

Corollary 13 (Ozawa). *We have the following* (1) *and* (2).

(1) *If* $1 \le p < 2$, *then* $\{(S_p^n(1), \|\cdot\|_2, \mu_{S_p^n(1)})\}_{n=1}^{\infty}$ *has the phase transition property with critical scale order* $\sim n^{\frac{1}{p}}$.

(2) *If* $2 \le p \le +\infty$, *then* $\{(S_p^n(1), \|\cdot\|_p, \mu_{S_p^n(1)})\}_{n=1}^{\infty}$ *has the phase transition property with critical scale order* $\sim n^{\frac{1}{p}}$. *In the case* $p = +\infty$, *we consider* $n^{\frac{1}{2p}} = 1$.

We consider the l_p-product mm-space $X_p^n = (X^n, d_{X_p^n}, \mu_X^{\otimes n})$, where $d_{X_p^n}$ is the l_p-product metric $d_{X_p^n}((x_1, \dots, x_n), (x_1', \dots, x_n')) := (\sum_{k=1}^n d_X(x_k, x_k')^p)^{1/p}$ for $(x_1, \dots, x_n)$, $(x_1', \dots, x_n') \in X^n$.

Theorem 14 (Ozawa-Shioya [13]). *Let p be an extended real number with $1 \le p \le +\infty$ and X a non-trivial mm-space with finite diameter. We assume* $\inf_{x, x' \in X, x \neq x'} d_X(x, x') > 0$ *if* $p \neq 1$. *Then we have*

$$\mathrm{ObsDiam}(X_p^n; -\kappa) \sim n^{\frac{1}{2p}}$$

for any positive real number κ with $0 < \kappa < 1$. In the case $p = +\infty$, we consider $n^{\frac{1}{2p}} = 1$. In particular, the sequence $\{X_p^n\}_{n=1}^{\infty}$ has the phase transition property with critical scale order $\sim n^{-\frac{1}{2p}}$.

We need the assumption $\inf_{x, x' \in X, x \neq x'} d_X(x, x') > 0$ if $p \neq 1$. In fact, we have the next proposition in the case of the l_2-product of compact and connected Riemannian manifold.

Proposition 15 (Gromov, Ozawa-Shioya [4, 13]). *Let M be a compact and connected Riemannian manifold with dimension at least one. Then the sequence $\{M_2^n\}_{n=1}^{\infty}$ of the l_2-product spaces has the phase transition property with critical scale order ~ 1.*

Let $M_\infty(\mathbb{R})$ be the set of real symmetric matrices of infinite order. For an mm-space X, define a map $K_\infty^X : X^\infty \to M_\infty(\mathbb{R})$ by $K_\infty^X(\{x_n\}_{n=1}^\infty) := (d_X(x_i, x_j))_{i,j=1}^\infty$. Define the *distance matrix distribution* $\underline{\mu}_\infty^X$ of X by $\underline{\mu}_\infty^X := (K_\infty^X)_* \mu_X^\infty$. Then $\underline{\mu}_\infty^X$ is a Borel probability measure on $M_\infty(\mathbb{R})$. The distance matrix distribution $\underline{\mu}_\infty^X$ is invariant under the action of the infinite permutation group $\mathfrak{S}_\infty$. This action is ergodic with respect to $\underline{\mu}_\infty^X$. For any positive real number λ, the convergence with respect to $\square_\lambda$ and the weak convergence of distance matrix distributions coincide with each other. Define a map $\tau : \mathcal{X} \to \mathcal{M}(M_\infty(\mathbb{R}))$ by $\tau(X) := \underline{\mu}_\infty^X$. In particular, the map τ is a topological embedding map with respect to $\square_1$ and the weak convergence. We consider to estimate the box distance between two mm-spaces X and X' by the two distance matrix distributions $\underline{\mu}_\infty^X$ and $\underline{\mu}_\infty^{X'}$. The Prohorov metric is a metrization of weak topology on the set of Borel probability measures on a separable metric space. In this thesis, we consider the Prohorov metric $d_P^{(M_\infty(\mathbb{R}), \|\cdot\|_\infty)}$ over a pseudo-metric space $(M_\infty(\mathbb{R}), \|\cdot\|_\infty)$. Moreover, we interpret the Prohorov distance between two distance matrix distributions as a new metric on $\mathcal{X}$.

Theorem 16 (Ozawa [10])**.** *Let X and X' be two mm-spaces. We define*

$$d_{l_\infty\text{-}P}(X,X') := d_P^{(M_\infty(\mathbb{R}),\|\cdot\|_\infty)}(\underline{\mu}_\infty^X, \underline{\mu}_\infty^{X'}).$$

Then $(\mathcal{X}, d_{l_\infty\text{-}P})$ is a metric space and we have

$$\square_1(X,X') \le d_{l_\infty\text{-}P}(X,X') \le \square_0(X,X').$$

Moreover, $(\mathcal{X},\square_1)$ and $(\mathcal{X}, d_{l_\infty\text{-}P})$ are not homeomorphic to each other by the identity map.

Define sets Met_∞ and $Met_\infty([\,0,1\,])$ by

$$\begin{aligned} Met_\infty &:= \{\ (d_{i,j})_{i,j=1}^\infty \mid d_{i,j} \ge 0,\ d_{i,i} = 0,\ d_{i,j} = d_{j,i},\ d_{i,j} \le d_{i,k} + d_{k,j}\ \}, \\ Met_\infty([\,0,1\,]) &:= \{\ (d_{i,j})_{i,j=1}^\infty \mid 0 \le d_{i,j} \le 1,\ d_{i,i} = 0,\ d_{i,j} = d_{j,i},\ d_{i,j} \le d_{i,k} + d_{k,j}\ \}. \end{aligned}$$

For an mm-space X, the distance matrix distribution $\underline{\mu}_\infty^X$ of X is an $\mathfrak{S}_\infty$-invariant Borel probability measure on Met_∞. This action is ergodic with respect to $\underline{\mu}_\infty^X$. Vershik suggested the following question to Gromov.

Question 17 (Vershik [4])**.** Let μ be an ergodic invariant Borel probability measure with respect to the $\mathfrak{S}_\infty$-action on Met_∞. Construct a generalized mm-space which represents μ as a generalized distance matrix distribution, and try to define basic mm-invariants for a generalized mm-space.

Elek [2] gave a partial answer of this question. For any $X \in \mathcal{X}_1$, the distance matrix distribution $\underline{\mu}_\infty^X$ of X is an $\mathfrak{S}_\infty$-invariant Borel probability measures on $Met_\infty([\,0,1\,])$. Since $Met_\infty([\,0,1\,])$ is a closed subset of $[\,0,1\,]^\infty$, the set $Met_\infty([\,0,1\,])$ is compact. Then, the set, say $\mathcal{M}(Met_\infty([\,0,1\,]))$, of Borel probability measures on $Met_\infty([\,0,1\,])$ equipped with the weak topology is compact. In particular, the closure $\overline{\tau(\mathcal{X}_1)}$ is also compact. He studied the closure $\overline{\tau(\mathcal{X}_1)}$ and proved that any element of $\overline{\tau(\mathcal{X}_1)}$ is expressed by a generalized mm-space called a qmm-space. The following is the definition of qmm-space. Denote by $\mathcal{M}([\,0,1\,])$ the set of Borel probability measures on $[\,0,1\,]$ equipped with the weak topology.

Definition 18 (qmm-Space; Elek [2])**.** A triple $Q = (Q, \mu_Q, d_Q^*)$ is called a *qmm-space* (*quantum metric measure space*) if it satisfies the following (1), (2), and (3).

(1) (Q, μ_Q) is a separable completely metrizable topological space with a Borel probability measure.

(2) $d_Q^* : Q \times Q \to \mathcal{M}([\,0,1\,])$ is a measurable map and satisfies $d_Q^*(q,q) = \delta_0$ a.s. $q \in Q$ and $d_Q^*(q,q') = d_Q^*(q',q)$ a.s. $(q,q') \in Q^2$.

(3) For any $t_{i,j} \in \operatorname{supp}(d_Q^*(q_i,q_j))$, $i,j = 1,2,3$, we have

$$t_{1,3} \le t_{1,2} + t_{2,3}$$

a.s. $(q_1,q_2,q_3) \in Q^3$.

Denote by $\mathcal{Q}_1$ the isomorphism classes of qmm-spaces.

For any mm-space X with diameter at most one, we define $d_X^*(x_1, x_2) := \delta_{d_X(x_1,x_2)}$, where $\delta_{d_X(x_1,x_2)}$ is the Dirac measure at $d_X(x_1, x_2) \in [0, 1]$. Then

$$Q_X := (X, \mu_X, d_X^*)$$

is a qmm-space. A convergence of a sequence of qmm-spaces is defined by a weak convergence of correspondence elements in $\overline{\tau(\mathcal{X}_1)}$. This convergence is called *convergence in sampling*. In particular, $\mathcal{Q}_1$ is a compactification of $(\mathcal{X}_1, \square_1)$. The idea of qmm-space came from the graph limit theory developed by Lovász-Szegedy [8].

In this thesis, we prove that there exists a metric on $\mathcal{Q}_1$ such that a map $\iota' : \mathcal{X}_1 \ni X \mapsto Q_X \in \mathcal{Q}_1$ is a 1-Lipschitz topological embedding map. This is an analogy of Shioya's result mentioned before.

Theorem 19 (Ozawa). *We have the following* (1) *and* (2).

(1) *There exists a metric υ on $\mathcal{Q}_1$ such that υ is a metrization of convergence in sampling.*

(2) *For any $X, X' \in \mathcal{X}_1$, we have*

$$\upsilon(Q_X, Q_{X'}) \leq \square(X, X').$$

In particular, ι' is a 1-Lipschitz embedding map with respect to $\square_1$ and υ.

Elek also generalized the observable diameter and the separation distance for qmm-space and considered the relation between the convergence in sampling, the observable diameter, and the separation distance.

Theorem 20 (Elek [2]). *Let Q and Q_n, $n = 1, 2, \ldots$, be qmm-spaces. If Q_n converges to Q in sampling as $n \to \infty$, then*

$$\lim_{\delta \to 0+} \limsup_{n \to \infty} \mathrm{ObsDiam}(Q_n; -(\kappa + \delta)) \leq \mathrm{ObsDiam}(Q; -\kappa)$$

for any positive real number κ with $0 < \kappa \leq 1$.

Theorem 21 (Elek [2]). *Let Q and Q_n, $n = 1, 2, \ldots$, be qmm-spaces. If Q_n converges to Q in sampling as $n \to \infty$, then*

$$\lim_{\delta \to 0+} \limsup_{n \to \infty} \mathrm{Sep}(Q_n; \kappa_0 - \delta, \kappa_1 - \delta, \ldots, \kappa_N - \delta) \leq \mathrm{Sep}(Q; \kappa_0, \kappa_1, \ldots, \kappa_N)$$

for any positive real numbers $\kappa_0, \kappa_1, \ldots, \kappa_N$ with $N \geq 1$.

We also generalize the concentration function for qmm-space and consider the relation between the convergence in sampling and the concentration function.

Theorem 22 (Ozawa [11]). *Let Q and Q_n, $n = 1, 2, \ldots$, be qmm-spaces. If Q_n converges to Q in sampling as $n \to \infty$, then*

$$\lim_{\delta \to 0+} \limsup_{n \to \infty} \alpha_{Q_n}(r - \delta, \kappa - \delta) \leq \alpha_Q(r, \kappa)$$

for any positive real numbers r and κ with $0 < \kappa, r \leq 1$.

Note that if Q and Q_n, $n = 1, 2, \ldots$, are mm-spaces, we have the equations in Theorem 20–22. We do not have the reverse inequalities in general. One of counterexamples for the reverse inequalities is the sequence $\{S^n(\pi^{-1})\}_{n=1}^{\infty}$ of n-dimensional spheres of radius π^{-1} equipped with the geodesic distance and the normalized Riemannian volume measure.

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